Fermionic Casimir effect in toroidally compactified de Sitter spacetime

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Abstract

We investigate the fermionic condensate and the vacuum expectation values of the energy-momentum tensor for a massive spinor field in de Sitter spacetime with spatial topology $\mathrm{R}^p \times (\mathrm{S}^1)^q$. Both cases of periodicity and antiperiodicity conditions along the compactified dimensions are considered. By using the Abel-Plana formula, the topological parts are explicitly extracted from the vacuum expectation values. In this way the renormalization is reduced to the renormalization procedure in uncompactified de Sitter spacetime. It is shown that in the uncompactified subspace the equation of state for the topological part of the energy-momentum tensor is of the cosmological constant type. Asymptotic behavior of the topological parts in the expectation values is investigated in the early and late stages of the cosmological expansion. In the limit when the comoving length of a compactified dimension is much smaller than the de Sitter curvature radius the topological part in the expectation value of the energy-momentum tensor coincides with the corresponding quantity for a massless field and is conformally related to the corresponding flat spacetime result. In this limit the topological part dominates the uncompactified de Sitter part. In the opposite limit, for a massive field the asymptotic behavior of the topological parts is damping oscillatory for both fermionic condensate and the energy-momentum tensor.

1 Introduction

In recent years much attention has been paid to the possibility that a universe could have non-trivial topology [1, 2]. Many of high energy theories of fundamental physics are formulated in higher dimensional spacetimes and it is commonly assumed that the extra dimensions are compactified. In particular, the idea of compactified dimensions has been extensively used in supergravity and superstring theories. From an inflationary point of view universes with compact spatial dimensions, under certain conditions, should be considered a rule rather than an exception [3]. The models of a compact universe with non-trivial topology may play an important role by providing proper initial conditions for inflation (for physical motivations of considering compact universes see also [4]). There has been a large activity to search for signatures of non-trivial topology by identifying ghost images of galaxies, clusters or quasars. Recent progress in observations of the cosmic microwave background provides an alternative way to observe the topology of the universe [2]. If the scale of periodicity is close to the particle horizon scale then

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the changed appearance of the microwave background sky pattern offers a sensitive probe of the topology.

The compactification of spatial dimensions leads to a number of interesting quantum field theoretical effects which include instabilities in interacting field theories [5], topological mass generation [6, 7, 8], symmetry breaking [8, 9]. In the case of non-trivial topology the boundary conditions imposed on fields give rise to the modification of the spectrum for vacuum fluctuations and, as a result, to the Casimir-type contributions in the vacuum expectation values of physical observables (for the topological Casimir effect and its role in cosmology see [10, 11] and references therein). Compactification of extra dimensions have moduli parameters which parametrize the size and the shape of the extra dimensions and the Casimir effect has been used to stabilize these moduli. The Casimir energy can also serve as a model for dark energy needed for the explanation of the present accelerated expansion of the universe (see [12, 13, 14] and references therein).

De Sitter (dS) spacetime is among the most important cosmological backgrounds. There are several physical motivations for this. In most inflationary models an approximately dS spacetime is employed to solve a number of problems in standard cosmology [15]. More recently astronomical observations of high redshift supernovae, galaxy clusters and cosmic microwave background [16] indicate that at the present epoch the universe is accelerating and can be well approximated by a world with a positive cosmological constant. If the universe would accelerate indefinitely, the standard cosmology would lead to an asymptotic dS universe. In addition to the above, an interesting topic which has received increasing attention is related to string-theoretical models of dS spacetime and inflation. Recently a number of constructions of metastable dS vacua within the framework of string theory are discussed (see, for instance, [17, 18] and references therein). dS spacetime is the maximally symmetric solution of the Einstein equations with a positive cosmological constant and due to its high symmetry numerous physical problems are exactly solvable on this background. A better understanding of physical effects in this background could serve as a handle to deal with more complicated geometries.

As it was argued in Refs. [19], there is no reason to believe that the version of dS spacetime which may emerge from string theory, will necessarily be the most familiar version with symmetry group $O(1,4)$ and there are many different topological spaces which can accept the dS metric locally. There are many reasons to expect that in string theory the most natural topology for the universe is that of a flat compact three-manifold. The quantum creation of the universe having toroidal spatial topology is discussed in [20] and in references [21] within the framework of various supergravity theories. The effect of the compactification of a single spatial dimension in dS spacetime (topology $\mathrm{R}^{D-1} \times \mathrm{S}^1$) on the properties of quantum vacuum for a scalar field with general curvature coupling parameter and with periodicity condition along the compactified dimension is investigated in Ref. [22] (for quantum effects in braneworld models with dS spaces and in higher-dimensional brane models with compact internal spaces see, for instance, Refs. [23, 24]). More general classes of the compactification with topology $\mathrm{R}^p \times (\mathrm{S}^1)^q$ in the case of scalar fields with both periodicity and antiperiodicity conditions are investigated in [25]. Note that for a scalar field on the background of uncompactified dS spacetime the renormalized vacuum expectation values of the field square and the energy-momentum tensor are investigated in Refs. [26, 27, 28] (see also [29]). The corresponding effects upon phase transitions in an expanding universe are discussed in [30, 31]. Trace anomaly for higher spin fields in dS spacetime is considered in Ref. [32].

In the present paper, partly motivated by possible applications in supersymmetric theories, we investigate one-loop quantum effects for a fermionic field on background of 4-dimensional dS spacetime with spatial topology $\mathrm{R}^p \times (\mathrm{S}^1)^q$. Among the most important quantities characterizing the properties of the fermionic vacuum are the fermionic condensate and the expectation value of

the energy-momentum tensor. Though the corresponding operators are local, due to the global nature of the vacuum, these quantities describe the global properties of the bulk and carry an important information about the topology. In addition, the vacuum expectation value of the energy-momentum tensor acts as a source of gravity in the Einstein equations and, hence, plays an important role in modelling self-consistent dynamics involving the gravitational field. We have organized the paper as follows. In the next section we consider the plane wave fermionic eigenfunctions for the problem under consideration. In sections 3 and 4 these eigenfunctions are used for the evaluation of the fermionic condensate and the vacuum expectation values of the energy-momentum tensor. The behavior of these quantities is investigated in various asymptotic regions of the parameters. In section 5, a fermionic field with antiperiodiciy conditions along the compactified dimensions (twisted field) is considered. The last section contains a summary of the work.

2 Plane wave eigenspinors in de Sitter spacetime with compact spatial dimensions

The dynamics of a massive fermionic field in curved spacetime with the metric tensor $g^{\mu\nu}$ is governed by the Dirac equation

$$i\gamma^\mu\nabla_\mu\psi - m\psi = 0 , \tag{1}$$

with the covariant derivative operator

$$\nabla_\mu = \partial_\mu + \Gamma_\mu . \tag{2}$$

Here $\gamma^\mu = e^\mu_{(a)}\gamma^{(a)}$ are the Dirac matrices in curved spacetime and Γ_μ is the spin connection given in terms of the flat space Dirac matrices $\gamma^{(a)}$ by the relation

$$\Gamma_\mu = \frac{1}{4}\gamma^{(a)}\gamma^{(b)}e^\nu_{(a)}e_{(b)\nu;\mu} . \tag{3}$$

Note that in this formula ; means the standard covariant derivative for vector fields. In the equations above $e^\mu_{(a)}$ is the tetrad field satisfying the relation $e^\mu_{(a)}e^\nu_{(b)}\eta^{ab} = g^{\mu\nu}$, where η^{ab} is the Minkowski spacetime metric tensor. In the discussion below the flat space Dirac matrices will be taken in the standard form

$$\gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma^{(a)} = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}, \ a = 1,2,3, \tag{4}$$

with $\sigma_1, \sigma_2, \sigma_3$ being the Pauli matrices.

We consider a quantum fermionic field on background of 4-dimensional de Sitter spacetime described by the line element

$$ds^2 = dt^2 - e^{2t/\alpha}\sum_{i=1}^3(dz^i)^2, \tag{5}$$

and having spatial topology $R^p \times (S^1)^q$, $p + q = 3$. The Ricci scalar and the corresponding cosmological constant are related to the parameter α in the expression for the scale factor by the formulae

$$R = 12/\alpha^2, \ \Lambda = 3/\alpha^2. \tag{6}$$

For the further discussion, in addition to the synchronous time coordinate t it is convenient to introduce the conformal time τ in accordance with

$$\tau = -\alpha e^{-t/\alpha}, \ -\infty < \tau < 0. \tag{7}$$

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In terms of this coordinate the line element takes the conformally flat form:

$$ds^2 = (\alpha/\tau)^2 [d\tau^2 - \sum_{i=1}^{3}(dz^i)^2]. \tag{8}$$

One of the characteristic features of field theory on backgrounds with non-trivial topology is the appearance of inequivalent types of fields with the same spin [33]. In particular, for fermion fields the boundary conditions along the compactified dimensions can be either periodic (untwisted field) or antiperiodic (twisted field). In this section we consider the field with periodicity conditions (no summation over l):

$$\psi(t, \mathbf{z}_p, \mathbf{z}_q + L_l \mathbf{e}_l) = \psi(t, \mathbf{z}_p, \mathbf{z}_q), \tag{9}$$

where $\mathbf{z}_p = (z^1, \ldots, z^p)$ and $\mathbf{z}_q = (z^{p+1}, \ldots, z^3)$ are the position vectors along uncompactified and compactified dimensions, $\mathbf{e}_l$ is the unit vector in the direction of the coordinate z^l with the length L_l, $0 \leqslant z^l \leqslant L_l$. The case of a fermionic field with antiperiodicity conditions will be discussed in section 5. The compactification of the spatial dimensions leads to the modification of the spectrum for zero point fluctuations of the fermionic field and as a result to change of the vacuum expectation values (VEVs) of physical observables. This is the well-known topological Casimir effect.

Among the most important quantities describing both local and global properties of the vacuum are the fermionic condensate and the VEV of the energy-momentum tensor. In order to evaluate these VEVs we expand the field operator in terms of a complete set of positive and negative frequency eigenspinors $\{\psi_\beta^{(+)}, \psi_\beta^{(-)}\}$:

$$\hat{\psi} = \sum_\beta [\hat{a}_\beta \psi_\beta^{(+)} + \hat{b}_\beta^+ \psi_\beta^{(-)}], \tag{10}$$

where $\hat{a}_\beta$ is the annihilation operator for particles, and $\hat{b}_\beta^+$ is the creation operator for antiparticles. The collective index β specifies the eigenfunctions. Next, we substitute expansion (10) and the similar expansion for the Dirac adjoint operator $\hat{\bar{\psi}} = \hat{\psi}^+ \gamma^0$ into the expressions for the bilinear product $\hat{\bar{\psi}}\hat{\psi}$ and the energy-momentum tensor,

$$T_{\mu\nu}\{\hat{\bar{\psi}}, \hat{\psi}\} = \frac{i}{2}[\hat{\bar{\psi}}\gamma_{(\mu}\nabla_{\nu)}\hat{\psi} - (\nabla_{(\mu}\hat{\bar{\psi}})\gamma_{\nu)}\hat{\psi}] . \tag{11}$$

By making use of the standard anticommutation relations for the annihilation and creation operators, one finds the following mode-sum formulae

$$\langle 0|\bar{\psi}\psi|0\rangle = \sum_\beta \bar{\psi}_\beta^{(-)}(x)\psi_\beta^{(-)}(x), \tag{12}$$

$$\langle 0|T_{\mu\nu}|0\rangle = \sum_\beta T_{\mu\nu}\{\bar{\psi}_\beta^{(-)}(x), \psi_\beta^{(-)}(x)\} , \tag{13}$$

where $|0\rangle$ is the amplitude for the vacuum state.

In order to find the form of the fermionic eigenfunctions for the geometry under consideration, we choose the basis tetrad in the form

$$e_\mu^{(0)} = \delta_\mu^0, \ e_\mu^{(a)} = e^{t/\alpha}\delta_\mu^a, \ a = 1, 2, 3. \tag{14}$$

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By using Eq. (14), for the components of the spin connection and for the combination appearing in the Dirac equation we find

$$\Gamma_l = \frac{e^{t/\alpha}}{2\alpha}\gamma^{(0)}\gamma^{(l)}, \ l = 1, 2, 3,$$

$$\Gamma_0 = 0, \ \gamma^\mu\Gamma_\mu = \frac{3}{2\alpha}\gamma^{(0)}. \tag{15}$$

In accordance with the symmetry of the problem under consideration the spatial part of the eigenfunctions can be taken in the standard plane wave form $e^{\pm i\mathbf{kr}}$, where $\mathbf{k}$ is the wave vector and the upper/lower sign corresponds to the positive/negative frequency solutions. We will decompose the wave vector into the components along the uncomcatified and compactified dimensions, $\mathbf{k} = (\mathbf{k}_p, \mathbf{k}_q)$. For a spinor field with periodicity conditions along the compactified dimensions one has

$$\mathbf{k}_q = (2\pi n_{p+1}/L_{p+1}, \ldots, 2\pi n_3/L_3), \tag{16}$$

with $n_{p+1}, \ldots, n_3 = 0, \pm 1, \pm 2, \ldots$.

By taking into account formulae (15), from equation (1) for the positive frequency solutions one finds

$$\left[\gamma^{(0)}\partial_t + ie^{-t/\alpha}(\mathbf{k}\cdot\boldsymbol{\gamma}) + 3\gamma^{(0)}/2\alpha + im\right]\psi = 0, \tag{17}$$

where $\boldsymbol{\gamma} = (\gamma^1, \gamma^2, \gamma^2)$. Further, we write the four-component spinor fields in terms of two-component ones as

$$\psi = \left(\begin{array}{c} \varphi_+ \\ \varphi_- \end{array}\right). \tag{18}$$

From equation (17) one finds the set of two first order differential equations for the functions $\varphi_\pm$:

$$(\eta\partial_\eta - 3/2 \mp i\alpha m)\,\varphi_\pm - i\eta(\mathbf{k}\cdot\boldsymbol{\sigma})\varphi_\mp = 0, \tag{19}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and we have introduced a new independent variable η in accordance with

$$\eta = \alpha e^{-t/\alpha}, \ 0 \leqslant \eta < \infty. \tag{20}$$

In accordance with (7) one has the simple relation $\eta = -\tau$ between this variable and the conformal time (note that we are still working in the coordinate system defined by Eq. (5)). Eqs. (19) lead to the following second order differential equations for the separate functions:

$$\left(\eta^2\partial_\eta^2 - 3\eta\partial_\eta + k^2\eta^2 + \alpha^2 m^2 \pm i\alpha m + 15/4\right)\varphi_\pm = 0, \tag{21}$$

where

$$k = |\mathbf{k}| = \sqrt{\mathbf{k}_p^2 + \mathbf{k}_q^2}. \tag{22}$$

The solutions of equations (21) for the functions $\varphi_\pm$ are linear combinations of the functions $\eta^2 H^{(1)}_{\pm 1/2 - i\alpha m}(k\eta)$ and $\eta^2 H^{(2)}_{\pm 1/2 - i\alpha m}(k\eta)$, with $H^{(1,2)}_\nu(x)$ being the Hankel functions. Different choices of the coefficients in these linear combinations correspond to different choices of the vacuum state. We will consider de Sitter invariant Bunch-Davies vacuum [28] for which the coefficients of the part containing the functions $H^{(2)}_{\pm 1/2 - i\alpha m}(k\eta)$ are zero. Hence, for a fermionic field in the Bunch-Davies vacuum the solutions to Eqs. (21) are the functions

$$\varphi_+ = \varphi^{(c)}\eta^2 H^{(1)}_{1/2 - i\alpha m}(k\eta), \ \varphi_- = -i\varphi^{(c)}(\mathbf{n}\cdot\boldsymbol{\sigma})\eta^2 H^{(1)}_{-1/2 - i\alpha m}(k\eta), \tag{23}$$

where $\varphi^{(c)}$ is an arbitrary constant spinor, $\mathbf{n} = \mathbf{k}/k$, and we have used Eq. (19) with the upper sign to express the function φ_- through the function φ_+.

On the base of solutions (23) we can construct the positive frequency solutions to the Dirac equation in the form (for the analog flat spacetime construction see, for instance, [34])

$$\psi_\beta^{(+)} = A_\beta \eta^2 e^{i\mathbf{k}\cdot\mathbf{r}} \begin{pmatrix} H_{1/2-i\alpha m}^{(1)}(k\eta)w^{(\sigma)} \\ -i(\mathbf{n}\cdot\boldsymbol{\sigma})H_{-1/2-i\alpha m}^{(1)}(k\eta)w^{(\sigma)} \end{pmatrix}, \tag{24}$$

where $\sigma = \pm 1/2$, $\beta = (\mathbf{k}, \sigma)$, and

$$w^{(1/2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ w^{(-1/2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{25}$$

In the similar way, for the negative frequency solutions we find

$$\psi_\beta^{(-)} = A_\beta \eta^2 e^{-i\mathbf{k}\cdot\mathbf{r}} \begin{pmatrix} i(\mathbf{n}\cdot\boldsymbol{\sigma})H_{-1/2+i\alpha m}^{(2)}(k\eta)w^{(\sigma)\prime} \\ H_{1/2+i\alpha m}^{(2)}(k\eta)w^{(\sigma)\prime} \end{pmatrix}, \tag{26}$$

with $w^{(\sigma)\prime} = 2i\sigma w^{(\sigma)}$.

The normalization coefficient A_β is determined from the orthonormalization condition

$$\int d^3x \sqrt{\gamma}\psi_\beta^{(\lambda)+}\psi_{\beta'}^{(\lambda')} = \delta_{\beta\beta'}\delta_{\lambda\lambda'}, \ \lambda, \lambda' = \pm, \tag{27}$$

with γ being the determinant of the spatial metric. On the right of this condition $\delta_{\beta\beta'}$ is understood as the Dirac delta function for continuous indices and the Kronecker delta for discrete ones. By making use of the Wronskian relation for the Hankel functions, one finds

$$A_\beta^2 = \frac{ke^{\pi\alpha m}}{2^{p+2}\pi^{p-1}V_q\alpha^3}, \tag{28}$$

where $V_q = L_{p+1}\cdots L_3$ is the volume of the compactified subspace. For a massless fermionic field we have the standard conformal relation $\psi_\beta^{(\pm)} = (\eta/\alpha)^{3/2}\psi_\beta^{(M)(\pm)}$ between eigenspinors (24), (26) defining the Bunch-Davies vacuum in dS spacetime and the corresponding eigenspinors $\psi_\beta^{(M)(\pm)}$ for the Minkowski spacetime with spatial topology $\mathrm{R}^p \times (\mathrm{S}^1)^q$. Note that the plane wave eigenspinors of the type (24) and (26) in uncompactified dS spacetime have been considered recently in Ref. [35].

3   Fermionic condensate

Substituting the eigenfunctions (26) into the mode-sum formula (12), for the fermionic condensate in dS spacetime with spatial topology $\mathrm{R}^p \times (\mathrm{S}^1)^q$ one finds

$$\begin{aligned} \langle\bar{\psi}\psi\rangle_{p,q} &= \frac{\eta^4 e^{\pi\alpha m}}{2^p \pi^{p/2-1}\Gamma(p/2)V_q\alpha^3} \int_0^\infty dk_p \, k_p^{p-1} \sum_{n_\mathbf{q}=-\infty}^{+\infty} k \\ &\times \left[H_{-1/2-i\alpha m}^{(1)}(k\eta)H_{-1/2+i\alpha m}^{(2)}(k\eta) - H_{1/2-i\alpha m}^{(1)}(k\eta)H_{1/2+i\alpha m}^{(2)}(k\eta) \right], \end{aligned} \tag{29}$$

where

$$\sum_{n_\mathbf{q}=-\infty}^{+\infty} = \sum_{n_{p+1}=-\infty}^{+\infty} \cdots \sum_{n_3=-\infty}^{+\infty}. \tag{30}$$

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The VEV given by formula (29) is divergent and needs some regularization procedure. To make it finite we can introduce a cut-off function $\varphi_\lambda(k)$ with the cut-off parameter λ, which decreases sufficiently fast with increasing k and satisfies the condition $\varphi_\lambda(k) \to 1$, for $\lambda \to 0$. As a next step we apply to the series over n_{p+1} the Abel-Plana summation formula [10, 36]

$$\sideset{}{'}\sum_{n=0}^{\infty} f(n) = \int_0^\infty dx\, f(x) + i \int_0^\infty dx\, \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1}, \tag{31}$$

where the prime means that the term $n = 0$ should be halved. The term in the VEV with the first integral in the right hand side of Eq. (31) corresponds to the fermionic condensate in dS spacetime with topology $\mathrm{R}^{p+1} \times (\mathrm{S}^1)^{q-1}$. As a result, the condensate is presented in the decomposed form

$$\langle \bar\psi\psi \rangle_{p,q} = \langle \bar\psi\psi \rangle_{p+1,q-1} + \Delta_{p+1} \langle \bar\psi\psi \rangle_{p,q}, \tag{32}$$

where the term

$$
\begin{aligned}
\Delta_{p+1} \langle \bar\psi\psi \rangle_{p,q} &= \frac{2^{4-p}\eta^4 \alpha^{-3}}{\pi^{p/2+1}\Gamma(p/2)V_{q-1}} \int_0^\infty dk_p\, k_p^{p-1} \sum_{n_{\mathbf{q-1}}=-\infty}^{+\infty} \int_0^\infty dx\, x^2 \\
&\quad \times \frac{\mathrm{Im}\left[K_{1/2-i\alpha m}(\eta x) I_{1/2+i\alpha m}(\eta x) \right]}{\sqrt{x^2 + k_p^2 + k_{\mathbf{n}_{q-1}}^2}\left(e^{L_{p+1}\sqrt{x^2+k_p^2+k_{\mathbf{n}_{q-1}}^2}} - 1\right)},
\end{aligned}
\tag{33}
$$

is induced by the compactness of the $(p+1)$th dimension. In Eq. (33), $I_\nu(x)$ and $K_\nu(x)$ are the Bessel modified functions, $V_{q-1} = L_{p+2} \cdots L_3$, and

$$k_{\mathbf{n}_{q-1}}^2 = \sum_{l=p+2}^{3} (2\pi n_l/L_l)^2. \tag{34}$$

Note that the expression on the right of formula (33) is finite and we have safely removed the cut-off function. Of course, we could expect the finiteness of the topological part as the toroidal compactification does not change the local geometry and, hence, the structure of the divergences is the same as in uncompactified dS spacetime.

An alternative form for the topological part of the fermionic condensate is obtained expanding the function $1/(e^y - 1)$ in the integrand and explicitly integrating over k_p. This leads to the result

$$
\begin{aligned}
\Delta_{p+1} \langle \bar\psi\psi \rangle_{p,q} &= \frac{32\eta^4 \alpha^{-3}}{(2\pi)^{(p+3)/2} V_{q-1}} \sum_{n=1}^{\infty} \sum_{n_{\mathbf{q-1}}=-\infty}^{+\infty} \int_0^\infty dx\, x^2 \\
&\quad \times \frac{\mathrm{Im}\left[K_{1/2-i\alpha m}(\eta x) I_{1/2+i\alpha m}(\eta x) \right]}{(nL_{p+1})^{p-1}} f_{(p-1)/2}\left(nL_{p+1}\sqrt{x^2 + k_{\mathbf{n}_{q-1}}^2}\right),
\end{aligned}
\tag{35}
$$

where we have introduced the notation

$$f_\nu(x) = x^\nu K_\nu(x). \tag{36}$$

As it follows from formula (35), for a massless spinor field the topological part of the fermionic condensate vanishes. From this formula we also see that the topological part depends on the variable η and the length scales L_l in the combinations L_l/η. Noting that $a(\eta)L_l$ is the comoving length with $a(\eta) = \alpha/\eta$ being the scale factor, we conclude that the topological part of the fermionic condensate is a function of comoving lengths of the compactified dimensions. When

the length of one of the compactified dimensions is large, $L_l \to \infty$, $l \geqslant p+2$, the main contribution into the sum over n_l in (35) comes from large values of n_l and we can replace the summation by the integration. The corresponding integral can be evaluated by using the formula [37]

$$\int_0^\infty dz \, f_{(p-1)/2}(a\sqrt{z^2+b^2}) = \sqrt{\frac{\pi}{2}} \frac{f_{p/2}(ab)}{a}, \tag{37}$$

and from Eq. (35) the formula is obtained for topology $\mathrm{R}^{p+1} \times (\mathrm{S}^1)^{q-1}$.

On the base of the recurrence relation (32) the fermionic condensate can be decomposed as

$$\langle \bar{\psi}\psi \rangle_{p,q} = \langle \bar{\psi}\psi \rangle_{\mathrm{dS}} + \langle \bar{\psi}\psi \rangle_c, \tag{38}$$

where the first term on the right is the condensate in uncompactified dS spacetime and the part

$$\langle \bar{\psi}\psi \rangle_c = \sum_{l=1}^q \Delta_{4-l} \langle \bar{\psi}\psi \rangle_{3-l,l}, \tag{39}$$

is induced by the non-trivial spatial topology.

Now we turn to the investigation of the topological part in the asymptotic regions of the parameters. First let us consider the limit $L_{p+1}/\eta \ll 1$. This corresponds to the limit when the comoving length of the $(p+1)$th direction is much smaller than the dS curvature radius: $a(\eta)L_{p+1} \ll \alpha$. Introducing a new integration variable $y = L_{p+1}x$ and by taking into account that for large values of u one has

$$\mathrm{Im}[I_{1/2+i\alpha m}(u)K_{1/2-i\alpha m}(u)] \sim -\frac{\alpha m}{2u^2}, \tag{40}$$

for the leading term in the asymptotic expansion of the fermionic condensate we find

$$\Delta_{p+1}\langle \bar{\psi}\psi \rangle_{p,q} \approx -\frac{8(\eta/\alpha)^2 m}{(2\pi)^{p/2+1}L_{p+1}^{p-1}V_q} \sum_{n=1}^\infty \sum_{n_{\mathbf{q-1}}=-\infty}^{+\infty} \frac{f_{p/2}(nL_{p+1}k_{\mathbf{n}_{q-1}})}{n^p}, \quad L_{p+1}/\eta \ll 1. \tag{41}$$

Hence, in the limit under consideration the topological part is negative. As due to the maximal symmetry of the dS spacetime the part $\langle \bar{\psi}\psi \rangle_{\mathrm{dS}}$ in the fermionic condensate is time-independent, we conclude that in this limit the topological part dominates. Taking into account the relation between η and the synchronous time coordinate, we see that formula (41) describes the asymptotic behavior in the early stages of the cosmological expansion corresponding to $t \to -\infty$.

In the opposite limit, when $\eta/L_{p+1} \ll 1$, we again introduce the new integration variable $y = L_{p+1}x$. By making use of the formulae for the Bessel modified functions for small values of the arguments, to the leading order we obtain

$$\Delta_{p+1}\langle \bar{\psi}\psi \rangle_{p,q} \approx -\frac{2^{3-p/2}\alpha B_0 e^{-4t/\alpha}}{\pi^{(p+1)/2}L_{p+1}^{p+1}V_q \cosh(\alpha m\pi)} \sin[2mt - 2\alpha m \ln(\alpha/L_{p+1}) - \phi_0], \tag{42}$$

where B_0 and ϕ_0 are defined by the relation

$$B_0 e^{i\phi_0} = \frac{2^{-i\alpha m}}{\Gamma(1/2+i\alpha m)} \sum_{n=1}^\infty \sum_{n_{\mathbf{q-1}}=-\infty}^{+\infty} \frac{f_{p/2+1+i\alpha m}(nL_{p+1}k_{\mathbf{n}_{q-1}})}{n^{p+2+2i\alpha m}}. \tag{43}$$

In terms of the synchronous time coordinate this limit corresponds to the late stages of the cosmological evolution, $t \to +\infty$. Hence, in the limit when the comoving length of the compactified dimensions is much larger than the curvature radius of dS spacetime, for a massive

fermionic field the topological part oscillates with exponentially decreasing amplitude with respect to the synchronous time coordinate. Note that the damping factor in the amplitude and the oscillation frequency are the same for all terms in the sum on the left hand side of formula (39) and, hence, we have the similar oscillating behavior for the total topological term: $\langle \bar{\psi}\psi \rangle_c \propto e^{-4t/\alpha} \sin{(2mt + \phi_c)}$.

In the special case of spatial topology $R^2 \times S^1$, from general formula (35) for the topological part in the fermionic condensate one finds

$$\langle \bar{\psi}\psi \rangle_c = -\frac{4\eta\alpha^{-3}}{\pi^2 L} \int_0^\infty dx\, x^2 \mathrm{Im}\left[K_{1/2-i\alpha m}(x)I_{1/2+i\alpha m}(x)\right]\ln(1-e^{-Lx/\eta}), \qquad (44)$$

where $L \equiv L_3$ is the length of the compactified dimension. On the left panel of figure 1 we have plotted this quantity for an untwisted fermionic field (dashed curves) as a function of L/η for the values of the parameter $\alpha m = 0.5, 1$ (the numbers near the curves). Writing $L/\eta = a(\eta)L/\alpha$, we see that this ratio is the comoving length of the compactified dimension in units of the dS curvature radius. As it follows from the asymptotic formula (42), for large values of the ratio
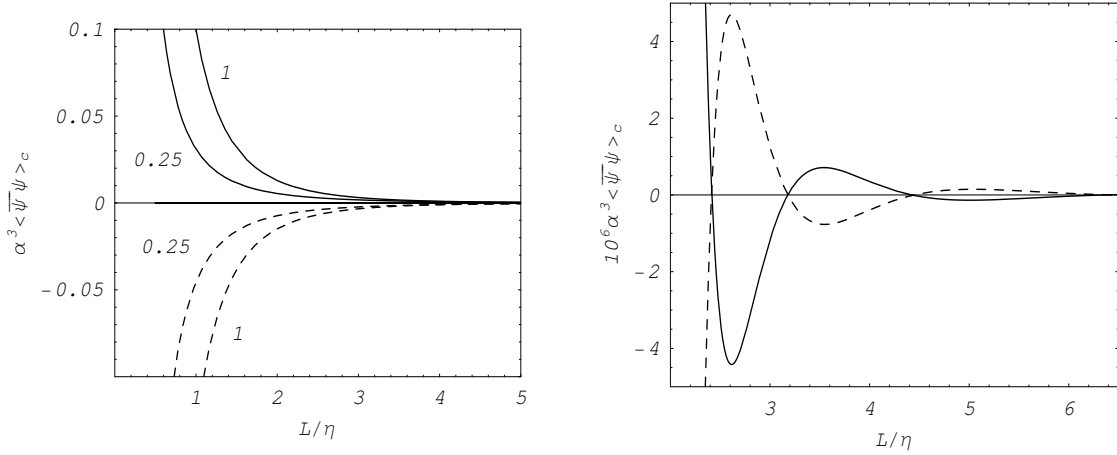

Figure 1: The topological part in the fermionic condensate for periodic (dashed curves) and antiperiodic (full curves) spinor fields in dS spacetime with spatial topology $R^2 \times S^1$ as a function of $L/\eta = Le^{t/\alpha}/\alpha$. The numbers near the curves on the left panel correspond to the values of the parameter αm. The graphs on the right panel are plotted for $\alpha m = 4$.

L/η the topological part behaves like $\langle \bar{\psi}\psi \rangle_c \propto (\eta/L)^4 \sin[2m\alpha \ln(\eta/L) + \phi_0]$. As in the scale of the left panel the oscillations are not well seen, we illustrate this oscillatory behavior on the right panel of figure 1, where the topological part in the fermionic condensate is plotted for an untwisted fermionic field by the dashed curve versus L/η for $\alpha m = 4$. The first zero with respect to L/η and the distance between neighbor zeros decrease with increasing values of the parameter αm.

4   Energy-momentum tensor

Now we turn to the investigation of the one-loop topological effects in the VEV of the energy-momentum tensor for a fermionic field with periodic boundary conditions. Substituting the eigenfunctions (26) into the mode-sum formula (13), for the energy density and vacuum stresses

we find (no summation over l)

$$
\begin{aligned}
\langle 0|T_0^0|0\rangle &= \frac{2^{-p}\eta^5\alpha^{-4}}{\pi^{p/2-1}\Gamma(p/2)V_q}\int_0^\infty dk_p\, k_p^{p-1}\sum_{n_{\mathbf{q}}=-\infty}^{+\infty}k^2 \\
&\quad\times\mathrm{Re}\left[H_{1/2+i\alpha m}^{(1)}(k\eta)H_{-1/2+i\alpha m}^{(2)\prime}(k\eta)-H_{-1/2+i\alpha m}^{(1)}(k\eta)H_{1/2+i\alpha m}^{(2)\prime}(k\eta)\right],\quad(45)
\end{aligned}
$$

$$
\langle 0|T_l^l|0\rangle = \frac{2^{1-p}\eta^5\alpha^{-4}}{\pi^{p/2-1}\Gamma(p/2)V_q}\int_0^\infty dk_p\, k_p^{p-1}\sum_{n_{\mathbf{q}}=-\infty}^{+\infty}k_l^2\mathrm{Re}\left[H_{1/2+i\alpha m}^{(1)}(k\eta)H_{1/2+i\alpha m}^{(2)}(k\eta)\right],\quad(46)
$$

with $l = 1,2,3$, and the off-diagonal components vanish. For a massless field, by using the expressions for the functions $H_{1/2}^{(1,2)}(x)$, we have the formula

$$
\langle 0|T_k^l|0\rangle = \frac{2(\eta/\alpha)^4}{(2\pi)^pV_q}\int d\mathbf{k}_p\sum_{n_{\mathbf{q}}=-\infty}^{+\infty}\frac{1}{k}\mathrm{diag}(-k^2,k_1^2,k_2^2,k_3^2).\quad(47)
$$

It can be easily seen that the expression on the right of this formula coincides with the leading term in the expansion of $\langle 0|T_k^l|0\rangle$ for a massive field in the limit $\eta\to\infty$. Note that this limit corresponds to the adiabatic limit of the slow expansion (for a detailed discussion see [29]).

As for the case of the fermionic condensate, in order to have finite expressions we will assume that in expressions (45), (46) a cut-off function is introduced. By using the Abel-Plana formula, the VEV of the energy-momentum tensor is presented in the decomposed form

$$
\langle T_k^l\rangle_{p,q} = \langle T_k^l\rangle_{p+1,q-1} + \Delta_{p+1}\langle T_k^l\rangle_{p,q}.\quad(48)
$$

In this formula, $\langle T_k^l\rangle_{p+1,q-1}$ is the VEV of the energy-momentum tensor for the topology $\mathrm{R}^{p+1}\times(\mathrm{S}^1)^{q-1}$ and the part $\Delta_{p+1}\langle T_k^l\rangle_{p,q}$ is due to the compactness of the $(p+1)$th dimension. The latter is given by the formula (no summation over l)

$$
\begin{aligned}
\Delta_{p+1}\langle T_l^l\rangle_{p,q} &= \frac{2^{2-p}\eta^5\alpha^{-4}}{\pi^{p/2}\Gamma(p/2)V_{q-1}}\int_0^\infty dk_p k_p^{p-1}\sum_{n_{\mathbf{q-1}}=-\infty}^{+\infty}\int_0^\infty dz \\
&\quad\times\frac{z h^{(l)}(z)F^{(l)}(\eta z)}{\sqrt{z^2+k_p^2+k_{\mathbf{n}_{q-1}}^2}(e^{L_{p+1}\sqrt{z^2+k_p^2+k_{\mathbf{n}_{q-1}}^2}}-1)},
\end{aligned}\quad(49)
$$

with the notations

$$
\begin{aligned}
h^{(0)}(z) &= -z^2,\ h^{(l)}(z) = k_p^2/p,\ l=1,\ldots,p, \\
h^{(p+1)}(z) &= -(z^2+k_p^2+k_{\mathbf{n}_{q-1}}^2),\ h^{(l)}(z) = k_l^2,\ l=p+2,3.
\end{aligned}\quad(50)
$$

In Eq. (54), to simplify the formulae we have introduced the notations

$$
F^{(0)}(y) = \frac{2}{\pi}\mathrm{Re}\left[I_{1/2+i\alpha m}(y)K'_{1/2-i\alpha m}(y)-K_{1/2-i\alpha m}(y)I'_{1/2+i\alpha m}(y)\right],\quad(51)
$$

$$
F^{(l)}(y) = \frac{\mathrm{Re}[I_{-1/2-i\alpha m}^2(y)-I_{1/2+i\alpha m}^2(y)]}{\cosh(\alpha m\pi)},\ l=1,2,3.\quad(52)
$$

The topological parts are finite and we have removed the cut-off functions in the corresponding expressions. Note that the following relations take place between the functions $F^{(l)}(y)$ and the

corresponding function appearing in the formula for the fermionic condensate:

$$\begin{aligned}
F^{(0)}(y) &= -F^{(1)}(y) - \frac{4\alpha m}{\pi y} \text{Im}\left[K_{1/2-i\alpha m}(y)I_{1/2+i\alpha m}(y)\right], \\
\partial_y[yF^{(0)}(y)] &= \frac{4\alpha m}{\pi y} \text{Im}\left[K_{1/2-i\alpha m}(y)I_{1/2+i\alpha m}(y)\right].
\end{aligned} \tag{53}$$

Similar to the case of the fermionic condensate, the alternative form for the topological parts is obtained expanding the integrand in Eq. (49) and explicitly evaluating the integral over k_p:

$$\begin{aligned}
\Delta_{p+1}\langle T_l^l\rangle_{p,q} &= \frac{4\eta^5\alpha^{-4}}{(2\pi)^{(p+1)/2}V_{q-1}} \sum_{n=1}^{\infty} \sum_{n_{\mathbf{q-1}}=-\infty}^{+\infty} \int_0^{\infty} dx\, x \\
&\quad \times \frac{F^{(l)}(\eta x)}{(L_{p+1}n)^{p+1}} f_p^{(l)}(nL_{p+1}\sqrt{x^2 + k_{\mathbf{n}_{q-1}}^2}),
\end{aligned} \tag{54}$$

where

$$\begin{aligned}
f_p^{(0)}(y) &= -(nL_{p+1}x)^2 f_{(p-1)/2}(y), \\
f_p^{(l)}(y) &= f_{(p+1)/2}(y),\ l = 1,\dots,p, \\
f_p^{(p+1)}(y) &= -\left[pf_{(p+1)/2}(y) + y^2 f_{(p-1)/2}(y)\right], \\
f_p^{(l)}(y) &= k_l^2(nL_{p+1})^2 f_{(p-1)/2}(y),\ l = p+2,\dots,3.
\end{aligned} \tag{55}$$

Note that we have the relation

$$\sum_{l=1}^{3} f_p^{(l)}(y) = f_p^{(0)}(y). \tag{56}$$

By using Eqs. (53), (56), we can see that the topological parts satisfy the standard trace relation

$$\Delta_{p+1}\langle T_l^l\rangle_{p,q} = m\Delta_{p+1}\langle\bar{\psi}\psi\rangle_{p,q}. \tag{57}$$

As an additional check of our calculations we can also see that the topological parts obey the continuity equation $\left(\Delta_{p+1}\langle T_l^k\rangle_{p,q}\right)_{;k} = 0$, which for the geometry under consideration takes the form

$$\left(\eta\partial_\eta - 4\right)\Delta_{p+1}\langle T_0^0\rangle_{p,q} + \Delta_{p+1}\langle T_l^l\rangle_{p,q} = 0. \tag{58}$$

This relation is proved by using formulae (53), (56) and (57).

From formulae (53) the relation $yF^{(1)}(y) = -(y^2 F^{(0)}(y))'$ is easily obtained. By making use of this relation and integrating by parts the expression (54) for $l = 1$, it can be seen that one has the relation (no summation over l)

$$\Delta_{p+1}\langle T_l^l\rangle_{p,q} = \Delta_{p+1}\langle T_0^0\rangle_{p,q},\ l = 1,\dots p, \tag{59}$$

between the energy density and the vacuum stresses along the uncompactified dimensions. In deriving (59) we have also used the formula $f_\nu'(z) = -zf_{\nu-1}$ which simply follows from well-known properties of the function $K_\nu(z)$. Hence, in the uncompactified subspace the equation of state for the topological part of the energy-momentum tensor is of the cosmological constant type. Note that the topological parts are time-dependent and they break the dS symmetry.

After the repetitive application of the recurrence formula (48), the VEV of the energy-momentum tensor for the topology $\mathrm{R}^p \times (\mathrm{S}^1)^q$ is presented in the form

$$\langle T_l^k\rangle_{p,q} = \langle T_l^k\rangle_{\mathrm{dS}} + \langle T_l^k\rangle_c,\ \langle T_l^k\rangle_c = \sum_{l=1}^{q} \Delta_{4-l}\langle T_l^k\rangle_{3-l,l}, \tag{60}$$

11

where $\langle T_l^k\rangle_{\mathrm{dS}}$ is the VEV in the uncompactified dS spacetime and $\langle T_l^k\rangle_c$ is the topological part. As the Bunch-Davies vacuum is dS invariant, the energy-momentum tensor $\langle T_l^k\rangle_{\mathrm{dS}}$ corresponds to a gravitational source of the cosmological constant type. In particular, combining with the initial cosmological constant Λ, one-loop effects in uncompactified dS spacetime lead to the effective cosmological constant $\Lambda_{\mathrm{eff}} = \Lambda + 8\pi G\langle T_0^0\rangle_{\mathrm{dS}}$, where G is the Newton's gravitational constant.

For a massless fermionic field, by using the expressions for the functions $I_{\pm 1/2}(y)$, $K_{1/2}(y)$, we see that

$$F^{(0)}(y) = -1/y, \ F^{(1)}(y) = 2/\pi y, \ m = 0. \tag{61}$$

Now the integration over x in (54) is done explicitly and one finds (no summation over l)

$$\Delta_{p+1}\langle T_l^l\rangle_{p,q} = \frac{8(\eta/\alpha)^4}{(2\pi)^{p/2+1}V_q L_{p+1}^{p+1}} \sum_{n=1}^{\infty} \sum_{\mathbf{n}_{q-1}=-\infty}^{+\infty} \frac{g_p^{(l)}(nL_{p+1}k_{\mathbf{n}_{q-1}})}{n^{p+2}}, \tag{62}$$

with the notations

$$\begin{aligned}
g_p^{(0)}(y) &= g_p^{(l)}(y) = f_{p/2+1}(y), \ l = 1,\ldots,p, \\
g_p^{(p+1)}(y) &= -(p+1)f_{p/2+1}(y) - y^2 f_{p/2}(y), \\
g_p^{(l)}(y) &= (nL_{p+1}k_l)^2 f_{p/2}(y), \ l = p+2,\ldots,3.
\end{aligned} \tag{63}$$

Note that in this case the problem under consideration is conformally related to the corresponding problem in the Minkowski spacetime with spatial topology $\mathrm{R}^p \times (\mathrm{S}^1)^q$ and formula (62) is obtained from the relation $\Delta_{p+1}\langle T_i^i\rangle_{p,q} = a^{-4}(\eta)\Delta_{p+1}\langle T_i^i\rangle_{p,q}^{(\mathrm{M})}$, with $a(\eta) = \alpha/\eta$ being the scale factor. Of course, this relation is valid for general case of the scale factor $a(\eta)$. Comparing expression (62) with the corresponding formula from [25] for a conformally coupled massless scalar field, we see that the following relation takes place: $\Delta_{p+1}\langle T_k^l\rangle_{p,q} = -4\Delta_{p+1}\langle T_k^l\rangle_{p,q}^{(\mathrm{scalar})}$. The latter is a simple generalization of the corresponding topologically non-trivial flat spacetime result (see, for instance, [32]). In curved backgrounds this relation between the topological parts of the fermionic and scalar energy-momentum tensors is not valid for a minimally coupled scalar field.

Let us consider the behavior of the vacuum energy-momentum tensor in the asymptotic regions of the parameters. In the limit $L_{p+1}/\eta \ll 1$, corresponding to small values of the comoving length $a(\eta)L_{p+1}$ with respect to the dS curvature radius, we introduce a new integration variable $y = L_{p+1}x$ in Eq. (54) and expand the functions $F^{(l)}(\eta y/L_{p+1})$. By taking into account that for large values z to the leading order

$$F^{(0)}(z) \approx -1/z, \ F^{(1)}(z) \approx 2/\pi z, \ z \gg 1, \tag{64}$$

we see that in this order the topological part in the VEV of the energy-momentum tensor coincides with that for a massless field given by formula (62). In particular, the topological part of the vacuum energy density is negative. Note that the limit under consideration corresponds to the early stages of the cosmological evolution, $t \to -\infty$. In this limit the topological parts in the VEVs of the energy-momentum tensors for periodic scalar and fermionic fields have opposite signs.

For small values of the ratio η/L_{p+1}, when the comoving length $a(\eta)L_{p+1}$ is larger that the dS curvature radius, after introducing the new integration variable $y = L_{p+1}x$, we see that the argument of the functions $F^{(l)}(\eta y/L_{p+1})$ is small. By making use of the formulae for the

Bessel modified functions for small arguments, in the leading order we obtain the formula (no summation over l)

$$\Delta_{p+1}\langle T_l^l\rangle_{p,q} \approx \frac{2^{2-p/2}B_l e^{-4t/\alpha}}{\pi^{(p+1)/2}V_q L_{p+1}^{p+1}\cosh(\alpha m\pi)}\cos[2mt - 2\alpha m\ln(\alpha/L_{p+1}) - \phi_l].\tag{65}$$

Here B_0, ϕ_0 are defined by relation (43), $B_l = B_0$, $\phi_l = \phi_0$ for $l = 1,\ldots,p$, and B_l, ϕ_l, $l = p+1,\ldots,3$, are defined by

$$
\begin{aligned}
B_{p+1}e^{i\phi_{p+1}} &= \frac{-2^{-i\alpha m}}{\Gamma(1/2+i\alpha m)}\sum_{n=1}^{\infty}\sum_{n_{\mathbf{q-1}}=-\infty}^{+\infty}\frac{1}{n^{p+2+2i\alpha m}}\\
&\quad \times \left[(p+1+2i\alpha m)f_{p/2+1+i\alpha m}(x) + x^2 f_{p/2+i\alpha m}(x)\right]_{x=nL_{p+1}k_{\mathbf{n}_{q-1}}},
\end{aligned}\tag{66}
$$

$$B_l e^{i\phi_l} = \frac{2^{-i\alpha m}}{\Gamma(1/2+i\alpha m)}\sum_{n=1}^{\infty}\sum_{n_{\mathbf{q-1}}=-\infty}^{+\infty}(L_{p+1}k_l)^2\frac{f_{p/2+i\alpha m}(nL_{p+1}k_{\mathbf{n}_{q-1}})}{n^{p+2i\alpha m}},\tag{67}$$

and in the last formula $l = p+2,\ldots,3$. Note that if the lengths of the compactified dimensions are the same the quantities B_l, ϕ_l do not depend on the compactification length. Formula (65) describes the asymptotic behavior of the topological part in the late stages of cosmological evolution which in terms of the synchronous time coordinate correspond to the limit $t \to +\infty$. As we see, in this limit the behavior of the topological part in the VEV of the energy-momentum tensor for a massive spinor field is damping oscillatory. This type of oscillations are absent in the case of a massless field when the topological parts behave like $e^{-4t/\alpha}$. For a massive field the damping factor in the amplitude and the oscillation frequency are the same for all terms in the sum of Eq. (60) and the total topological term behaves like $\langle T_l^k\rangle_c \propto e^{-4t/\alpha}\cos(2mt + \phi_c')$. As the vacuum energy-momentum tensor for uncompactified dS spacetime is time-independent, we have similar damping oscillations in the total energy-momentum tensor $\langle T_l^k\rangle_{\mathrm{dS}} + \langle T_l^k\rangle_c$ as well.

The general formulae for the topological part in the VEV of the energy-momentum tensor are further simplified in the special case of topology $\mathrm{R}^2 \times \mathrm{S}^1$. For this case the functions $f_p^{(l)}(x)$ are expressed in terms of exponentials and after the summation over n we find (no summation over l)

$$\langle T_l^l\rangle_c = \frac{(\eta/L)^3}{\pi\alpha^4}\int_0^{\infty}dx\, x F^{(l)}(x)G_l(Lx/\eta),\tag{68}$$

with $L = L_3$ being the length of the compactified dimension. In Eq. (68) the following notations are introduced

$$
\begin{aligned}
G_0(y) &= y^2\ln(1 - e^{-y}),\\
G_1(y) &= G_2(y) = y\mathrm{Li}_2(e^{-y}) + \mathrm{Li}_3(e^{-y}),\\
G_3(y) &= G_0(y) - 2G_1(y),
\end{aligned}\tag{69}
$$

and $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty}z^k/k^n$ is the polylogarithm function. For a massless fermionic field we have

$$\langle T_k^l\rangle_c = \frac{2\pi^2}{45}\left(\frac{\eta}{\alpha L}\right)^4\mathrm{diag}(1,1,1,-3).\tag{70}$$

In figure 2 by the dashed curves we have presented the dependence of the topological part in the VEV of the energy-momentum tensor (68) on the ratio L/η for an untwisted fermionic field in the case of topology $\mathrm{R}^2 \times \mathrm{S}^1$. The numbers near the curves correspond to the values of the index l. Note that, as we have seen above, the stresses along uncompactified dimensions ($l = 1, 2$) coincide with the energy density.
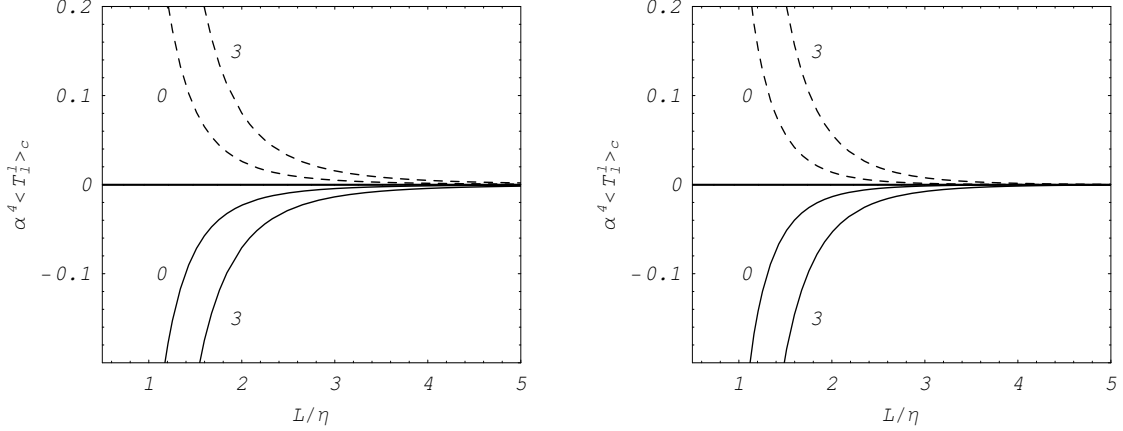
Figure 2: The topological parts in the VEV of the energy density ($l = 0$) and the stress along the compactified dimension ($l = 3$) for periodic (dashed curves) and antiperiodic (full curves) spinor fields in dS spacetime with spatial topology $\mathrm{R}^2 \times \mathrm{S}^1$ as functions of $L/\eta = L e^{t/\alpha}/\alpha$. The numbers near the curves correspond to the values of the index l. The vacuum stresses along the uncompactified dimensions are equal to the energy density. For the left panel $\alpha m = 0.25$ and for the right one $\alpha m = 1$.

As it has been shown before, for a massive field the topological part oscillates for large values of L/η. For an untwisted fermionic field this is shown separately in figure 3, where the energy density ($l = 0$) is plotted by the dashed curve for the value of the parameter $\alpha m = 4$. As in the case of the fermionic condensate, the first zero and the distance between the neighbor zeros decrease with increasing αm.

5 Spinor field with antiperiodicity conditions

In this section we consider the spinor field satisfying the antiperiodicity condition along the compactified dimensions

$$\psi(t, \mathbf{z}_p, \mathbf{z}_q + L_l \mathbf{e}_l) = -\psi(t, \mathbf{z}_p, \mathbf{z}_q). \tag{71}$$

The corresponding eigenfunctions are given by formulae (24), (26), where now the wave vector along the compactified diemnsions is given by the formula

$$\mathbf{k}_q = (\pi(2n_{p+1} + 1)/L_{p+1}, \ldots, \pi(2n_3 + 1)/L_3), \tag{72}$$

and $n_{p+1}, \ldots, n_3 = 0, \pm 1, \pm 2, \ldots$. The fermionic condensate and the VEV of the energy-momentum tensor are still given by formulae (45), (46) with

$$k^2 = k_p^2 + \sum_{l=p+1}^{3} \left[\pi(2n_l + 1)/L_l\right]^2. \tag{73}$$

Further, we apply to the series over n_{p+1} the Abel-Plana formula in the form [10, 36]

$$\sum_{n=0}^{\infty} f(n + 1/2) = \int_0^{\infty} dx\, f(x) - i \int_0^{\infty} dx\, \frac{f(ix) - f(-ix)}{e^{2\pi x} + 1}. \tag{74}$$
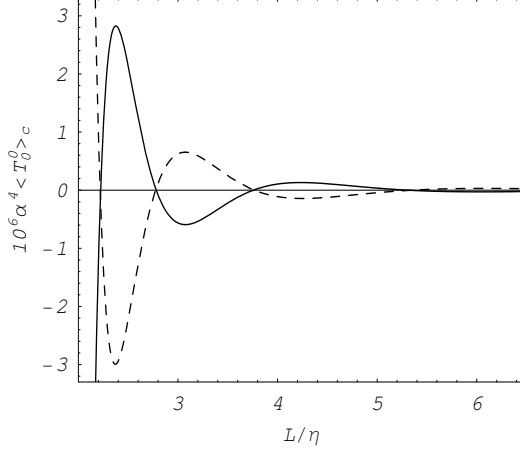
14

Figure 3: The topological parts in the VEV of the energy density for periodic (dashed curve) and antiperiodic (full curves) spinor fields in dS spacetime with spatial topology $R^2 \times S^1$ as functions of L/η. The graphs are plotted for the value of the parameter $\alpha m = 4$.

As a result, the topological part in the fermionic condensate is presented in the form (32), where now the part induced by the compactness of the $(p+1)$th direction is given by the formula

$$
\begin{aligned}
\Delta_{p+1}\langle\bar\psi\psi\rangle_{p,q} &= -\frac{4\eta^4\alpha^{-3}}{2^{p-1}\pi^{p+1}V_{q-1}}\int_0^\infty dk_p\, k_p^{p-1}\sum_{\mathbf{n}_{q-1}=-\infty}^{+\infty}\int_0^\infty dz \\
&\times z^2\frac{\mathrm{Im}\left[K_{1/2-i\alpha m}(\eta z)I_{1/2+i\alpha m}(\eta z)\right]}{\sqrt{z^2+k_p^2+k_{\mathbf{n}_{q-1}}^2}(e^{L_{p+1}\sqrt{z^2+k_p^2+k_{\mathbf{n}_{q-1}}^2}}+1)}.
\end{aligned}
\tag{75}
$$

with the notation

$$
k_{\mathbf{n}_{q-1}}^2 = \sum_{l=p+2}^{3}\left[\pi(2n_l+1)/L_l\right]^2.
\tag{76}
$$

Expanding the integrand and performing the integration over k_p, this part can also be presented in the form

$$
\begin{aligned}
\Delta_{p+1}\langle\bar\psi\psi\rangle_{p,q} &= \frac{32\eta^4\alpha^{-3}}{(2\pi)^{(p+3)/2}V_q L_{p+1}^{p-2}}\sum_{n=1}^\infty(-1)^n\sum_{\mathbf{n}_{q-1}=-\infty}^{+\infty}\int_0^\infty dx\, x^2 \\
&\times\frac{\mathrm{Im}\left[K_{1/2-i\alpha m}(\eta x)I_{1/2+i\alpha m}(\eta x)\right]}{n^{p-1}}f_{(p-1)/2}(nL_{p+1}\sqrt{x^2+k_{\mathbf{n}_{q-1}}^2}),
\end{aligned}
\tag{77}
$$

where the function $f_\nu(y)$ is defined by Eq. (36). For a massless field the topological part vanishes. Having the parts induced by separate compact dimensions we can decompose the fermionic condensate as given in Eqs. (38), (39).

General formula (77) is simplified in the special case of topology $R^2 \times S^1$:

$$
\langle\bar\psi\psi\rangle_c = -\frac{4\eta\alpha^{-3}}{\pi^2 L}\int_0^\infty dx\, x^2\mathrm{Im}\left[K_{1/2-i\alpha m}(x)I_{1/2+i\alpha m}(x)\right]\ln(1+e^{-Lx/\eta}),
\tag{78}
$$

where L is the length of the compact direction. On the left panel of figure 1 by full curves we have plotted the dependence of this quantity on the ratio L/η. The numbers near the curves

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correspond to the values of the parameter αm. The full curve on the right panel of figure 1 shows the oscillatory behavior of the fermionic condensate for a twisted field. The curve is plotted for the value $\alpha m = 4$.

In the similar way, for the topological parts in the VEV of the energy-momentum tensor we find the formula (no summation over l)

$$
\begin{aligned}
\Delta_{p+1}\langle T_l^l\rangle_{p,q} &= \frac{4\eta^5\alpha^{-4}}{(2\pi)^{(p+1)/2}V_qL_{p+1}^p}\sum_{n=1}^{\infty}(-1)^n\sum_{n_{\mathbf{q-1}}=-\infty}^{+\infty}\int_0^{\infty}dx \\
&\times x\frac{F^{(l)}(\eta x)}{n^{p+1}}f_p^{(l)}(nL_{p+1}\sqrt{x^2+k_{\mathbf{n}_{q-1}}^2}).
\end{aligned}
\tag{79}
$$

Hence, in the case of a fermionic field with antiperiodicity conditions the formulae for the topological parts in the VEVs are obtained from those for the field with periodicity conditions inserting the factor $(-1)^n$ in the summation over n and replacing the definition for $k_{\mathbf{n}_{q-1}}^2$ by (76). Note that, as in the case of periodic boundary conditions, we have relation (59) between the energy density and the vacuum stresses along the uncompactified dimensions.

For the case of topology $R^2 \times S^1$, in the way similar to that for periodic field, the formulae (79) are simplified giving (no summation over l)

$$
\langle T_l^l\rangle_c = \frac{(\eta/L)^3}{\pi\alpha^4}\int_0^{\infty}dx\,xF^{(l)}(x)G_l^{(\mathrm{tw})}(Lx/\eta),
\tag{80}
$$

with the notations

$$
\begin{aligned}
G_0^{(\mathrm{tw})}(y) &= y^2\ln(1+e^{-y}), \\
G_l^{(\mathrm{tw})}(y) &= y\mathrm{Li}_2(-e^{-y}) + \mathrm{Li}_3(-e^{-y}),\ l=1,2, \\
G_3^{(\mathrm{tw})}(y) &= G_0^{(\mathrm{tw})}(y) - 2G_l^{(\mathrm{tw})}(y).
\end{aligned}
\tag{81}
$$

For a massless field we have the result

$$
\langle T_k^l\rangle_c = -\frac{7\pi^2}{180}\left(\frac{\eta}{\alpha L}\right)^4\mathrm{diag}(1,1,1,-3).
$$

In figure 2 (full curves) the topological part in the VEV of the energy-momentum tensor (80) for a twisted fermionic field is plotted versus L/η. The numbers near the curves correspond to the values of the index l. As it has been shown before, for a massive field the topological parts oscillate for large values of L/η. The oscillations of the topological part in the energy density as a function of L/η are illustrated in figure 3 by the full curve plotted for the value of parameter $\alpha m = 4$.

6   Conclusion

Motivated by the importance of compactified dimensions in physical models and continuing our work [22, 25], in the present paper we have investigated the fermionic condensate and the VEV of the energy-momentum tensor for a massive fermionic field in dS spacetime with toroidally compactified spatial dimensions. In order to evaluate the corresponding mode-sums we need the eigenspinors satisfying appropriate boundary conditions along the compactified dimensions. For a fermionic field with periodicity conditions these eigenspinors are constructed in section 2 and are given by formulae (24), (26). Choosing these functions we have assumed that the field is in the Bunch-Davies vacuum state. The application of the Abel-Plana formula allows

us to present the VEVs for the spatial topology $R^p \times (S^1)^q$ as the sum of the corresponding quantity in the topology $R^{p+1} \times (S^1)^{q-1}$ and of the part which is induced by the compactness of $(p+1)$th dimension. The latter is finite and in this way the renormalization is reduced to that for the uncompactified dS spacetime. In the case of an untwisted field the topological parts in the fermionic condensate and in the VEV of the energy-momentum tensor are given by formulae (35), (54). The parts induced by the non-trivial topology are time-dependent and breake the dS symmetry. The corresponding vacuum stresses along the uncompactified dimensions coincide with the energy density and, hence, in the uncompactified subspace the equation of state for the topological part of the energy-momentum tensor is of the cosmological constant type. As an additional check of the formulae, we have shown that the topological parts satisfy trace relation (57) and the covariant continuity equation (58). For a massless fermionic field the problem under consideration is conformally related to the corresponding problem in the Minkowski spacetime with spatial topology $R^p \times (S^1)^q$ and the fermionic condensate vanishes. In this case we have the relation $\langle T_k^l \rangle_c = a^{-4}(\eta) \langle T_k^l \rangle_c^{(M)}$ between the topological contributions in the VEV of the energy-momentum tensor (see formula (62)). For a massless field the topological part in the VEV of the fermionic energy-momentum tensor differs from the corresponding quantity for a conformally coupled massless scalar field by an additional factor -4.

For the case of a massive field the general formulae are simplified in the asymptotic regions of the parameters. In the limit when the comoving length of a compactified dimension is much smaller than the dS curvature radius, $L_{p+1}/\eta \ll 1$, for an untwisted field to the leading order the topological part in the fermionic condensate is given by formula (41) and is negative. In the same limit, the topological part in the VEV of the energy-momentum tensor coincides with the corresponding quantity for a massless field and is conformally related to the corresponding VEV in toroidally compactified Minkowski spacetime. In particular, the vacuum energy density is negative. This limit corresponds to the early stages of the cosmological evolution and the topological parts dominate over the uncompactified dS parts. In the opposite limit, when the comoving lengths of the compactified dimensions are large with respect to the dS curvature radius, $L_{p+1}/\eta \gg 1$, in the case of a massive field the asymptotic behavior of the topological parts is damping oscillatory for both fermionic condensate and the energy-momentum tensor and to the leading order is given by formulae (42), (65). These formulae describe the behavior of the topological parts in the late stages of the cosmological expansion. As the uncompactified dS parts are time-independent, we have similar oscillations in the total VEVs as well. Note that this type of oscillatory behavior is absent for a massless fermionic field.

In section 5 we have considered a fermionic field with antiperiodicity conditions along the compactified dimensions. In this case the topological parts in the fermionic condensate and the VEV of the energy-momentum tensor are given by formulae (77), (79). The asymptotic expressions for these VEVs in the early and late stages of the cosmological expansion are obtained from the corresponding formulae for an untwisted field inserting the factor $(-1)^n$ in the summation over n and defining $k_{\mathbf{n}_{q-1}}^2$ in accordance with Eq. (76). The features in the behavior of the topological VEVs we have illustrated in figures 1-3 for both untwisted and twisted fermionic fields in the case of spatial topology $R^2 \times S^1$. For this topology the general expressions are further simplified to formulae (44), (68), (78), (80).

7   Acknowledgments

References

[1] M. Lachièze-Rey and J.-P. Luminet, Phys. Rep. **254**, 135 (1995).

[2] J. Levin, Phys. Rep. **365**, 251 (2002); N.J. Cornish, D.N. Spergel, G.D. Starkman, and E. Komatsu, Phys. Rev. Lett. **92**, 201302 (2004).

[3] A. Linde, JCAP **0410**, 004 (2004).

[4] G.D. Starkman, Class. Quantum Grav. **15**, 2529 (1998); N.J. Cornish, D.N. Spergel, and G.D. Starkman, Class. Quantum Grav. **15**, 2657 (1998).

[5] L.H. Ford, Phys. Rev. D **22**, 3003 (1980).

[6] L.H. Ford and T. Yoshimura, Phys. Lett. A **70**, 89 (1979).

[7] D.J. Toms, Phys. Rev. D **21**, 928 (1980).

[8] D.J. Toms, Phys. Rev. D **21**, 2805 (1980).

[9] S.D. Odintsov, Sov. J. Nucl. Phys. **48**, 1148 (1988); I.L. Buchbinder and S.D. Odintsov, Int. J. Mod. Phys. **A4**, 4337 (1989); Fortschr. Phys. **37**, 225 (1989); I.L.Buchbinder, S.D. Odintsov, and V.P. Dergaleo, Theor. Math. Phys. **80**, 150 (1989).

[10] V.M. Mostepanenko and N.N. Trunov, *The Casimir Effect and Its Applications* (Clarendon, Oxford, 1997).

[11] M. Bordag, U. Mohidden, and V.M. Mostepanenko, Phys. Rep. **353**, 1 (2001); K.A. Milton, *The Casimir Effect: Physical Manifestation of Zero-Point Energy* (World Scientific, Singapore, 2002); E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, *Zeta regularization techniques with applications* (World Scientific, Singapore, 1994); M.J. Duff, B.E.W. Nilsson, and C.N. Pope, Phys. Rep. **130**, 1 (1986); A.A. Bytsenko, G. Cognola, L. Vanzo, and S. Zerbini, Phys. Rep. **266**, 1 (1996).

[12] K.A. Milton, Grav. Cosmol. **9**, 66 (2003).

[13] E. Elizalde, J. Phys. A **39**, 6299 (2006).

[14] B. Green and J. Levin, JHEP **11**, 096 (2007); P. Burikham, A. Chatrabhuti, P. Patcharamaneepakorn, and K. Pimsamarn, arXiv:0802.3564.

[15] A.D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic Publishers, Chur, Switzerland 1990).

[16] A.G. Riess et al., Astrophys. J. **659**, 98 (2007); D.N. Spergel et al., Astrophys. J. Suppl. Ser. **170**, 377 (2007); U. Seljak, A. Slosar, and P. McDonald, JCAP **0610**, 014 (2006); E. Komatsu et al., arXiv:0803.0547.

[17] S. Kachru, R. Kallosh, A. Linde, and S.P. Trivedi, Phys. Rev. D **68**, 046005 (2003).

[18] E. Silverstein, arXiv:0712.1196.

[19] B. McInnes, Nucl. Phys. B **709**, 213 (2005); B. McInnes, Nucl. Phys. B **748** , 309 (2006).

[20] Y.B. Zeldovich and A.A. Starobinsky, Sov. Astron. Lett. **10**, 135 (1984).

[21] Yu.P. Goncharov and A.A. Bytsenko, Phys. Lett. B **160**, 385 (1985); Yu.P. Goncharov and A.A. Bytsenko, Phys. Lett. B **169**, 171 (1986); Yu.P. Goncharov and A.A. Bytsenko, Nucl. Phys. B **271**, 726 (1986); Yu.P. Goncharov and A.A. Bytsenko, Class. Quant. Grav. **4**, 555 (1987).

[22] A.A. Saharian and M.R. Setare, Phys. Lett. B **659**, 367 (2008).

[23] S. Nojiri, S. D. Odintsov, and S. Zerbini, Class. Quantum. Grav. **17**, 4855 (2000); W. Naylor and M. Sasaki, Phys. Lett. B **542**, 289 (2002); E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Ogushi, Phys. Rev. D **67**, 063515 (2003); I.G. Moss, W. Naylor, W. Santiago-German, and M. Sasaki, Phys. Rev. D **67**, 125010 (2003); K. Uzawa, Prog. Theor. Phys. **110**, 457 (2003); A. A. Saharian and M. R. Setare, Phys. Lett. B **584**, 306 (2004); M.R. Setare, Phys. Lett. B **620**, 111 (2005); M. Minamitsuji, W. Naylor, and M. Sasaki, Nucl. Phys. B **737**, 121 (2006); M.R. Setare, Phys. Lett. B **637**, 1 (2006).

[24] A. Flachi, J. Garriga, O. Pujolàs, and T. Tanaka, J. High Energy Phys. **08**, 053 (2003); A. Flachi and O. Pujolàs, Phys. Rev. D **68**, 025023 (2003); A.A. Saharian, Phys. Rev. D **73**, 044012 (2006); A.A. Saharian, Phys. Rev. D **73**, 064019 (2006); A.A. Saharian, Phys. Rev. D **74**, 124009 (2006).

[25] S. Bellucci, A.A. Saharian, arXiv:0802.2190.

[26] P. Candelas and D.J. Raine, Phys. Rev. D **12**, 965 (1975).

[27] J.S. Dowker and R. Critchley, Phys. Rev. D **13**, 224 (1976); J.S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976).

[28] T.S. Bunch and P.C.W. Davies, Proc. R. Soc. London **A360**, 117 (1978).

[29] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[30] A. Vilenkin and L.H. Ford, Phys. Rev. D **26**, 1231 (1982).

[31] B. Allen, Nucl. Phys. B **226**, 228 (1983).

[32] N.D. Birrell, J. Phys. A **12**, 337 (1979).

[33] C.J. Isham, Proc. R. Soc. London **A362**, 383 (1978); C.J. Isham, Proc. R. Soc. London **A364**, 591 (1978).

[34] V.B. Berestetskii, E.M. Lifshits, and L.P. Pitaevskii, *Quantum Electrodynamics* (Pergamon, Oxford, 1982).

[35] I.I. Cotăescu, Phys. Rev. D **65**, 084008 (2002).

[36] A. A. Saharian, "The Generalized Abel-Plana Formula with Applications to Bessel Functions and Casimir Effect," Report No. IC/2007/082; arXiv:0708.1187.

[37] A. P. Prudnikov, Yu.A. Brychkov, and O. I. Marichev, *Integrals and Series, Vol. 2: Special Functions* (Gordon and Breach, New York, 1986).